A FOUR PARAMETER GENERALIZATION OF GÖLLNITZ'S (BIG) PARTITION THEOREM

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Dedicated to our friend Barry McCoy on his sixtieth birthday

Abstract. We announce a new four parameter partition theorem from which the (big) theorem of Göllnitz follows by setting any one of the parameters equal to 0. This settles a problem of Andrews who asked whether there exists a result that goes beyond the partition theorem of Göllnitz. We state a four parameter q-series identity (key identity) which is the generating function form of this theorem. In a subsequent paper, the proof of the new four parameter key identity will be given.

§1. Introduction

Our purpose here is to announce the following new partition theorem:

Theorem 1. *Let $P(n)$ denote the number of partitions of n into distinct parts $\equiv -2^3, -2^2, -2^1, -2^0$ (mod 15).*

Let $G(n)$ denote the number of partitions of n into parts $\not\equiv 2^0, 2^1, 2^2, 2^3$ (mod 15) such that the difference between the non-multiples of 15 is ≥ 15 with equality only if a part is relatively prime to 15, parts which are not relatively prime to 15 are > 15, the difference between the multiples of 15 is ≥ 60, and the smallest multiple of 15 is

$$\geq 30 + 30\tau, \text{ if 7 is a part, and}$$
$$\geq 45 + 30\tau, \text{ otherwise},$$

where τ is the number of non-multiples of 15 in the partition. Then

$$G(n) = P(n).$$

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While it is obvious that this result is a partition theorem of the Schur-Göllnitz type, it is not clear that it lies beyond the (big) theorem of Göllnitz [15]. It is possible to obtain a four parameter refinement of Theorem 1; this is stated as Theorem 2 in §2. From this it follows that Theorem 1 (and Theorem 2) generalize the Göllnitz theorem in the same sense that Göllnitz's theorem extends Schur's 1926 partition theorem [16]. Thus the question raised by Andrews [12] nearly 30 years ago whether there exists a partition theorem that goes beyond the (big) theorem of Göllnitz, is now answered in the affirmative.

Our purpose here is only to announce the new results and describe how they extend Göllnitz's theorem. The proof of Theorem 2 (and consequently of Theorem 1) will be given in full in a subsequent paper [6].

In 1995, Alladi, Andrews, and Gordon [5] obtained a three parameter refinement of Göllnitz's theorem by the use of colored partitions. Because colors were labelled by letters and the integers occurring in the colors were indicated by subscripts (weights), this approach was called *the method of weighted words*. In §2 we will describe an extension of this method that leads to the new four parameter Theorem 2. In doing so some essentially new ideas are required. In §2 we will also state the generating function form of Theorem 2 which we call a four parameter *key identity*. By setting any one of the parameters equal to 0 in Theorem 2, we get the three parameter refinement of Göllnitz's theorem due to Alladi-Andrews-Gordon (see §3). Similarly by setting any one of the parameters equal to 0 in the new key identity (2.6), the three parameter key identity for Göllnitz's theorem in [5] falls out. Finally in §4 we conclude with a brief description of some problems for future research opened up by Theorem 2.

The two parameter key identity for Schur's theorem due to Alladi and Gordon [9] is essentially equivalent to the $q-$Chu-Vandermonde summation as shown by Alladi and Berkovich [8]. The three parameter key identity for Göllnitz's theorem due to Alladi, Andrews, and Gordon [5] that extends the identity in [9] is substantially deeper, and its proof utilizes either the ${}_6\psi_6$ summation of Bailey as in [5], or Jackson's $q-$analog of Dougall's summation as in [4]. The proof of the new four parameter key identity (2.6) also relies on the ${}_6\psi_6$ summation, but requires several new ideas and is significantly deeper than the proofs in [4] and [5]. That is why the proof of the new four parameter identity will be presented separately [6].

§2. Colored reformulation and a four parameter refinement

We consider the integer 1 occurring in four primatry colors $\mathbb{A}, \mathbb{B}, \mathbb{C}$, and $\mathbb{D}$, and integers $n \geq 2$ occurring in these four primary colors as well as in the six secondary colors $\mathbb{AB}, \mathbb{AC}, \mathbb{AD}, \mathbb{BC}, \mathbb{BD}$, and $\mathbb{CD}$. We assume that integers $n \geq 4$ occur in the

quaternary color $\mathbb{ABCD}$ in addition to the ten colors above. The crucial thing is that we discard all ternary colors $\mathbb{ABC}, \mathbb{ABD}, \mathbb{ACD}$, and $\mathbb{BCD}$.

The integer n in color $\mathbb{A}$ is denoted by the symbol $\mathbb{A}_n$ with similar interpretation for $\mathbb{B}_n, \ldots, \mathbb{CD}_n, \mathbb{ABCD}_n$. In order to discuss colored partitions, we need an ordering among the symbols, and so we assume that

$$\begin{cases} \text{ if } m < n \text{ as ordinary (uncolored) integers,} \\ \text{then } m \text{ in any color} < n \text{ in any color, and} \\ \text{if two equal integers appear in different colors, the order is given by} \\ \mathbb{ABCD} < \mathbb{AB} < \mathbb{AC} < \mathbb{AD} < \mathbb{A} < \mathbb{BC} < \mathbb{BD} < \mathbb{B} < \mathbb{CD} < \mathbb{C} < \mathbb{D}. \end{cases} \tag{2.1}$$

Next, given any partition into parts occurring in the eleven colors above, we denote by a the number of parts in color $\mathbb{A}$ (the frequency of $\mathbb{A}$), with similar interpretation for b, c, and d. Pursuing the same notation, ab will denote the number of parts in color $\mathbb{AB}$, with similar interpretation for $ac, \ldots, cd$. <u>Note that ab is not a times b!</u> Finally, Q denotes the number of parts in color $\mathbb{ABCD}$.

We are now in a position to state our main result..

Theorem 2. *Let i, j, k, l be given nonnegative integers.*

Let $P(n; i, j, k, l)$ denote the number of partitions of n into parts occurring in the four primary colors, parts in the same color being distinct, and with i parts in color $\mathbb{A}$, j parts in color $\mathbb{B}$, k parts in color $\mathbb{C}$, and l parts in color $\mathbb{D}$.

Let $G(n; a, b, c, d, ab, \ldots, cd, Q)$ denote the number of partitions of n into colored parts occurring in the frequencies as indicated, and such that the difference between the nonquaternary parts is ≥ 1, with equality only if parts are either of the same primary color, or if the larger part occurs in a color of higher order as given in (2.1), and the gap between the quaternary parts is ≥ 4, with the added condition that the least quaternary part is

$$\begin{aligned} &\geq 3 + 2\tau, \;\; \textit{if } \mathbb{A}_1 \textit{ is a part,} \\ &\geq 4 + 2\tau, \;\; \textit{otherwise,} \end{aligned} \tag{2.2}$$

where τ is the number of nonquaternary parts. Then

$$P(n; i, j, k, l) = \sum_{\text{constraints}} G(n; a, b, c, d, ab, \ldots, cd, Q). \tag{2.3}$$

where the summation is over the variables $a, b, ..., cd, Q$ satisfying the constraints

$$\begin{aligned} i &= a + ab + ac + ad + Q \\ j &= b + ab + bc + bd + Q \\ k &= c + ac + bc + cd + Q \\ l &= d + ad + bd + cd + Q. \end{aligned} \tag{2.4}$$

A strong four parameter refinement of Theorem 1 follows from Theorem 2 upon replacing

$$\begin{cases} \mathbb{A}_n \mapsto 15n-8, \mathbb{B}_n \mapsto 15n-4, \mathbb{C}_n \mapsto 15n-2, \mathbb{D}_n \mapsto 15n-1, \text{ for } n \geq 1, \\ \text{and consequently } \mathbb{AB}_n \mapsto 15n-12, \mathbb{AC}_n \mapsto 15n-10, \mathbb{AD}_n \mapsto 15n-9, \\ \mathbb{BC}_n \mapsto 15n-6, \mathbb{BD}_n \mapsto 15n-5, \mathbb{CD}_n \mapsto 15n-3, \text{ for } n \geq 2, \\ \text{and } \mathbb{ABCD}_n \mapsto 15n-15, \text{ for } n \geq 4. \end{cases} \tag{2.5}$$

The nice thing about the substitutions (2.5) is that the ordering (2.1) becomes

$$7 < 11 < 13 < 14 < 18 < 20 < 21 < 22 < 24 < 25 < 26 < 27 < 28 < 29 < 33 <, \ldots$$

the natural ordering among the integers $\not\equiv 2^0, 2^1, 2^2, 2^3 \pmod{15}$. These substitutions imply that the primary colors correspond to the residue classes

$$-2^3, -2^2, -2^1, -2^0 (\text{mod} \quad 15).$$

Since

$$2^0 + 2^1 + 2^2 + 2^3 = 15,$$

the ternary colors correspond to the residue classses

$$2^0, 2^1, 2^2, 2^3 (\text{mod} \quad 15).$$

These are the four residue classes not considered in Theorem 1. Also, since the residue classes relatively prime to 15 are $2^0, 2^1, 2^2, 2^3, -2^3, -2^2, -2^1, -2^0 \pmod{15}$, it follows that the secondary colors correspond to the nonmultiples of 15 which are not relatively prime to 15. Finally, the quaternary color corresponds to the multiples of 15 which are ≥ 45. These features make Theorem 1 particularly appealing.

The generating function form of Theorem 2 is the following remarkable four parameter *key identity*: If $T_n = n(n+1)/2$, and $\tau = a+b+c+d+ab+\cdots+cd$, then

$$\begin{aligned} &\sum_{\text{constraints}} \frac{q^{T_\tau + T_{ab} + \cdots + T_{cd} - bc - bd - cd + 4T_{Q-1} + 3Q + 2Q\tau}}{(q)_a (q)_b (q)_c (q)_d (q)_{ab} \cdots (q)_{cd} (q)_Q} \times \\ &\left\{ (1-q^a) + q^{a+bc+bd+Q}(1-q^b) + q^{a+bc+bd+Q+b+cd} \right\} \\ &= \frac{q^{T_i+T_j+T_k+T_l}}{(q)_i (q)_j (q)_k (q)_l}, \end{aligned} \tag{2.6}$$

where the constraints are as in (2.4) and the summation is over $a, b, \dots, Q$. In (2.6) we have made use of the standard notation

$$(A)_n = (A;q)_n = \begin{cases} \prod_{j=0}^{n-1}(1-Aq^j), & if \quad n>0, \\ 1, & if \quad n=0, \\ \prod_{j=1}^{-n}(1-Aq^{-j})^{-1}, & if \quad n<0. \end{cases}$$

In [6] we do not prove Theorem 2 combinatorially. Instead we prove (2.6) using $q-$series techniques and show that (2.6) is equivalent to Theorem 2.

§3. Reduction to Göllnitz and Schur

If any one of the parameters i, j, k, l, is set equal to 0, Theorem 2 reduces to Theorem 2 of [5], the three parameter refinement of the colored version of Göllnitz's theorem. Note that in this case, the quaternary color does not occur at all. We state the result with $l = 0$.

Theorem A. *Let $P(n; i, j, k)$ denote the number of partitions of n into parts occuring in three primary colors $\mathbb{A}, \mathbb{B}, \mathbb{C}$, parts in the same color being distinct, with i parts in color $\mathbb{A}$, j parts in color $\mathbb{B}$, and k parts in color $\mathbb{C}$.*

Let $G(n; a, b, c, ab, ac, bc)$ denote the number of partitions of n into parts occuring in colors $\mathbb{A}, \mathbb{B}, \mathbb{C}, \mathbb{AB}, \mathbb{AC}, \mathbb{BC}$, with indicated frequencies $a, b, ..., bc$, such that the difference between the parts is ≥ 1 with equality only if parts are either of the same primary color, or if the larger part occurs in a color of higher order as indicated in (2.1). Then

$$\sum_{\substack{i=a+ab+ac \\ j=b+ab+bc \\ k=c+ac+bc.}} G(n; a, b, c, ab, ac, bc) = P(n; i, j, k).$$

In Theorem A, if we use the substitutions

$$\begin{cases} \mathbb{A}_n \mapsto 6n-4, \mathbb{B}_n \mapsto 6n-2, \mathbb{C}_n \mapsto 6n-1, \text{ for } n \geq 1, \\ \text{yielding} \quad \mathbb{AB} \mapsto 6n-6, \mathbb{AC}_n \mapsto 6n-5, \mathbb{BC}_n \mapsto 6n-3, \text{ for } n \geq 2, \end{cases} \tag{3.1}$$

we get a three parameter refinement of the following theorem of Göllnitz [15].

Theorem G. *Let $P(n)$ denote the number of partitions of n into distinct parts $\equiv -2^2, -2^1, -2^0$ (mod 6).*

Let $G(n)$ denote the number of partitions of n into parts $\neq 1$ or 3, such that the difference between the parts ≥ 6, with equality only if a part is $\equiv -2^2, -2^1, -2^0$ (mod 6). Then

$$G(n) = P(n).$$

If any one of the parameters i, j, k, l is set equal to 0, then (2.6) reduces to the three parameter key identity for Göllnitz's theorem in [5]. For example, with $l = 0$, (2.6) reduces to

$$\sum_{\substack{i=a+ab+ac\\ j=b+ab+bc\\ k=c+ac+bc}} \frac{q^{T_\tau+T_{ab}+T_{ac}+T_{bc-1}}\{1-q^a+q^{a+bc}\}}{(q)_a(q)_b(q)_c(q)_{ab}(q)_{ac}(q)_{bc}} = \frac{q^{T_i+T_j+T_k}}{(q)_i(q)_j(q)_k}. \tag{3.2}$$

Here $\tau = a+b+c+ab+ac+bc$. If we further set one of i, j, k equal to 0, say $k = 0$, in (3.2), then we get the two parameter key identity for the colored version of Schur's theorem due to Alladi and Gordon [9], namely,

$$\sum_{\substack{i=a+ab\\ j=b+ab}} \frac{q^{T_{a+b+ab}+T_{ab}}}{(q)_a(q)_b(q)_{ab}} = \frac{q^{T_i+T_j}}{(q)_i(q)_j} \tag{3.3}$$

In extending the refined Schur theorem in [9] to the refined Göllnitz theorem in [5], the statement of the extension was routine once the theorems were phrased in the language of primary and secondary colors. The principal reason for the increase in difficulty in going up from the Schur theorem to the Göllnitz theorem is because in the lexicographic ordering, one of the secondary colors $\mathbb{BC}$ is of higher order than the primary color $\mathbb{A}$. In addition, even though the refined Göllnitz theorem uses three primary colors, the ternary color $\mathbb{ABC}$ is dropped and so only a proper subset of the complete alphabet of colors is used. Thus the Göllnitz theorem is an extension of Schur's theorem in a direction different from the one taken by Andrews [10], [11], who in retrospect used a complete alphabet of colors. In going beyond the Göllnitz theorem to Theorem 2, there is a significant increase in depth and complexity for a variety of reasons. The quaternary color enters in a rather unusual way - the quaternaries do not directly interact with primaries and secondaries. The only interaction between quaternaries and the other colors is through the lower bound imposed on the quaternaries. This, combined with the uncertainity of selecting a proper subset of colors from the complete alphabet of four primaries, six secondaries, four ternaries, and one quaternary, was perhaps the reason that the solution to the problem of Andrews [12] remained elusive for so long.

§4. Problems for investigation

If the expressions in (2.6) are multiplied of $A^iB^jC^kD^l$ and summed over i, j, k, l,

we get on the right hand side the quadruple infinite product

$$\prod_{m=1}^{\infty}(1+Aq^m)(1+Bq^m)(1+Cq^m)(1+Dq^m), \tag{4.1}$$

with four free parameters A, B, C, D. This opens up several avenues of exploration a few of which we briefly indicate here.

Theorem 2 may be viewed as a base level undilated version of Theorem 1. More precisely, the generating function form of Theorem 1 may be viewed as emerging out of (2.6) and (4.1) under the transformation

$$\begin{cases} \text{(dilation) } q \mapsto q^{15}, \\ \text{(translations) } A \mapsto Aq^{-8}, B \mapsto Bq^{-4}, C \mapsto Cq^{-2}, D \mapsto Dq^{-1}. \end{cases} \tag{4.2}$$

The size of the modulus 15 and the choice of the translations involving powers of 2 ensures that the colors in Theorem 2 translate into distinct residue classes mod 15. If a dilation smaller than $q \mapsto q^{15}$ is used, then the residue classes would overlap, and so we would be counting parts with weights attached. Alladi [1] has studied weighted partition identities in general and discussed certain interesting reformulations of Göllnitz's theorem and their applications [2], [3], emerging out of *small* dilations of (3.2). In a similar spirit it would be worthwhile to study weighted partition theorems emerging out of (2.6) by the use of dilations $q \mapsto q^M$, with $M < 15$. We anticipate obtaining new and different versions of partition theorems that have arisen in the study of affine Lie algebras, and representations of symmetric groups, by such weighted reformulations of Theorem 1.

Recently, Alladi and Berkovich [7] have obtained the following double bounded version of (3.3):

$$\sum_{k\geq 0} q^{T_{i+j-k}+T_k} \begin{bmatrix} M-i-j+k \\ k \end{bmatrix} \begin{bmatrix} M-j \\ i-k \end{bmatrix} \begin{bmatrix} L-i \\ j-k \end{bmatrix} = \begin{bmatrix} L \\ j \end{bmatrix} \begin{bmatrix} M-j \\ i \end{bmatrix} q^{T_i+T_j}. \tag{4.3}$$

In (4.3), the symbols $\begin{bmatrix} n+m \\ n \end{bmatrix}$ are the $q-$binomial coefficients given by

$$\begin{bmatrix} n+m \\ n \end{bmatrix}_q = \begin{bmatrix} n+m \\ n \end{bmatrix} = \begin{cases} \frac{(q^{m+1})_n}{(q)_n}, & if \quad n \geq 0, \\ 0, & if \quad n < 0. \end{cases}$$

If we let $L, M \to \infty$ in (4.3) and make the identifications $a = i-k$, $b = j-k$, and $ab = k$, then we get (3.3).

In addition, Alladi and Berkovich have also obtained a double bounded version of (3.2), namely,

$$\sum_{constraints} q^{T_\tau+T_{ab}+T_{ac}+T_{bc-1}}$$
$$\left\{q^{bc}\begin{bmatrix} L-\tau+a \\ a \end{bmatrix}\begin{bmatrix} L-\tau+b \\ b \end{bmatrix}\begin{bmatrix} M-\tau+c \\ c \end{bmatrix}\begin{bmatrix} L-\tau \\ ab \end{bmatrix}\begin{bmatrix} M-\tau \\ ac \end{bmatrix}\begin{bmatrix} M-\tau \\ bc \end{bmatrix}\right.$$
$$\left.+\begin{bmatrix} L-\tau+a-1 \\ a-1 \end{bmatrix}\begin{bmatrix} L-\tau+b \\ b \end{bmatrix}\begin{bmatrix} M-\tau+c \\ c \end{bmatrix}\begin{bmatrix} L-\tau \\ ab \end{bmatrix}\begin{bmatrix} M-\tau \\ ac \end{bmatrix}\begin{bmatrix} M-\tau \\ bc-1 \end{bmatrix}\right\}$$
$$=\sum_{t\geq 0} q^{t(M+2)-T_t+T_{i-t}+T_{j-t}+T_{k-t}}\begin{bmatrix} L-t \\ t \end{bmatrix}\begin{bmatrix} L-2t \\ i-t \end{bmatrix}\begin{bmatrix} L-i-t \\ j-t \end{bmatrix}\begin{bmatrix} M-i-j \\ k-t \end{bmatrix}, \tag{4.4}$$

where the constraints are as in (3.2) and $\tau = a+b+c+ab+ac+bc$. The proof (4.4), which is quite intricate, is given in [8]. If we let $L, M \to \infty$, then (3.2) follows because only the term corresponding to $t = 0$ on the right hand side in (4.4) makes a contribution. Based on the discovery of (4.4), we now ask whether there exists a finite bounded version of (2.6) (which reduces to (2.6) when certain parameters tend to infinity).

In the last decade, many new generalizations of the Rogers-Ramanujan identities were discovered and proved by McCoy and collaborators (see [14] for a review and references), using the so called thermodynamic Bethe ansatz (TBA) techniques. It would be highly desirable to find a TBA interpretation of the new identity (2.6). Such an interpretation, besides being of substantial interest in physics, may provide insight into how to extend Theorem 2 to five or more primary colors.

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